The University of Nottingham

DEPARTMENT OF Mechanical Engineering

A LEVEL 2 MODULE, Spring SEMESTER 2020-2021

MMME2046 Dynamics and Controls

Time allowed 3 hours plus 30 minutes upload period

Open-book take-home examination

Answer ALL questions

You must submit a single pdf document, produced in accordance with the guidelines provided on take-home examinations, that contains all of the work that you wish to have marked for this open-book examination. Your submission file should be named in the format '[Student ID]_[Module Code].pdf'.

Write your student ID number at the top of each page of your answers.

This work must be carried out and submitted as described on the Moodle page for this module. All work must be submitted via Moodle by the submission deadline. **Work submitted after the deadline will not be accepted without a valid EC.**

No academic enquiries will be answered by staff and no amendments to papers will be issued during the examination. If you believe there is a misprint, note it in your submission but answer the question as written.
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Question 1.

Figure Q1 shows a rigid bar of length L that slides down. Its point A is sliding along a horizontal plane and B is sliding along an inclined plane ($\beta = 60°$).

1) How many degrees of freedom does the system have at the instant shown? (1 points)
2) Find the angle α at the moment when the amplitudes of velocities of the points A and B are equal. (4 points)

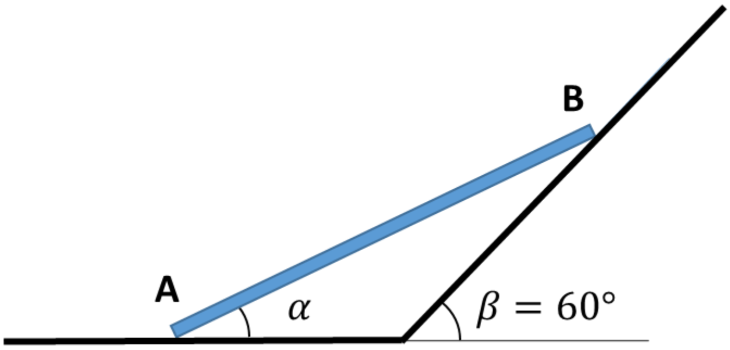
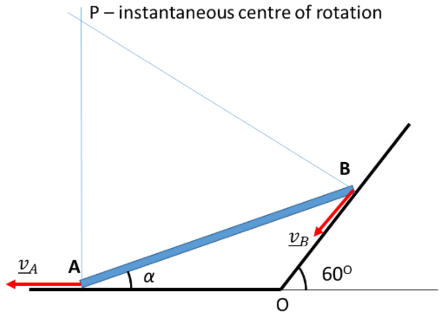


Figure Q1.

Solution:

1) This system has one degree of freedom – angle between the bar and the wall (until the rigid bar touches both walls). **+1**
2) Two methods to solve this problem:
   Using instantaneous centre of rotation



$\omega = PA \cdot v_A = PB \cdot v_B$ **+2**

Therefore, PA=PB when $v_A = v_B$. This means ∠PAB=∠PBA (isosceles triangle) and ∠BAO=∠ABO=α (also isosceles triangle). Therefore, from triangle ABO

$\alpha + \alpha + 120° = 180°$ $\alpha = 30°$ **+2**

Or using Chasles theorem (projection on AB):

$v_A \cos(\alpha) = v_B \cos(\angle ABO)$ **+2**

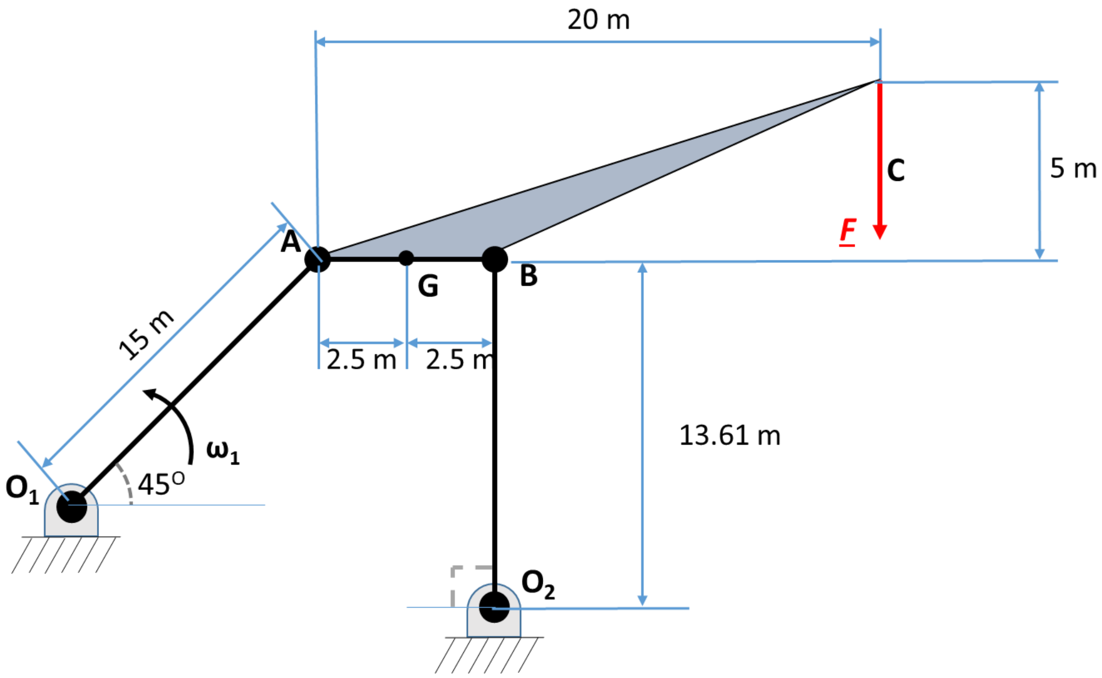
$\cos(\alpha) = \cos(\angle ABO)$ when $v_A = v_B$. Therefore, ∠ABO=α. From triangle ABO:

$\alpha + \alpha + 120° = 180°$ $\alpha = 30°$ **+2**

Question 2.

Figure 2 shows a four-link mechanism (a crane mechanism) with a load F=10^4 N applied to point C in the vertical direction. Points O_1 and O_2 are fixed. The lengths of the links are: O_1A = 15 m, AB = 5m, O_2B=13.61 m. Link O_1A rotates with constant angular velocity ω_1=2 rad/s. Link ABC has mass concentrated along edge AB. The centre of mass of link ABC is located at point G and AG=2.5 m. The mass of ink ABC is 1000 kg, and its moment of inertia is 5000 kg.m^2.

1) Find the velocity of point C at the instant shown. (5 points)
2) Find the acceleration of point G at the instant shown. (5 points)
3) Perform dynamic force analysis on the mechanism in the instant shown by using D'Alembert's Principle:
   a) Draw the free body diagram of the mechanism; (5 points)
   b) Find the torque applied to the link O_1A required to drive the mechanism. Ignore gravity forces in all links and assume that links O_1A and O_2B are massless; (5 points)
   c) Find the reaction forces at points O_1 and O_2. (5 points)



1) Velocity analysis

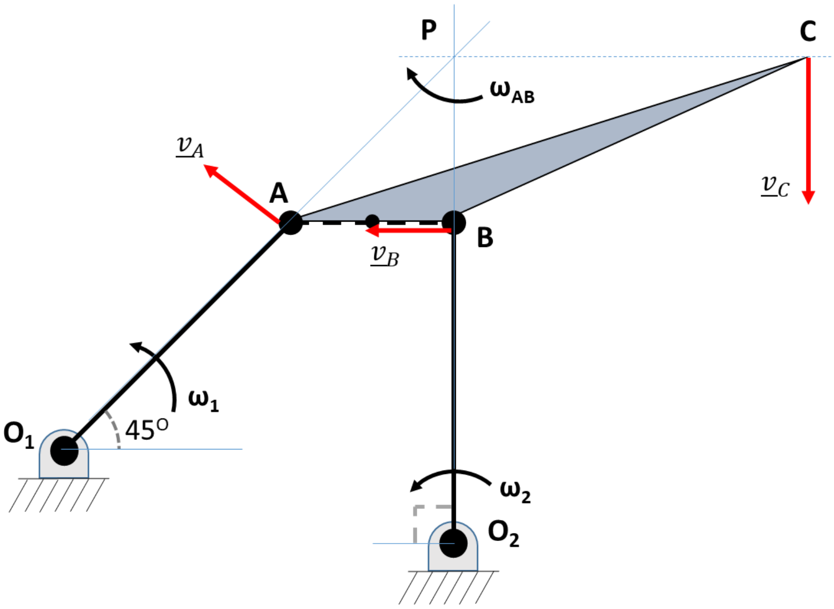
Using instantaneous centre of rotation

+1 point for the correct sketch

Triangle PAB is isosceles with a right angle. Therefore, PB=5 m, PA=7.07 m.

$v_A = \omega_1 O_1A = 30$ m/s, then, $\omega_{AB} = v_A/PA = 4.24$ rad/s (**+1 point**) and $v_B = \omega_{AB}PB = 21.21$ m/s (**+1 point**), $\omega_2 = v_B/O_2B = 1.56$ rad/s (**+1 point**).

Finally, $v_C = \omega_{AB}PC = 4.24 \cdot 15 = 63.65$ m/s (**+1 point**).



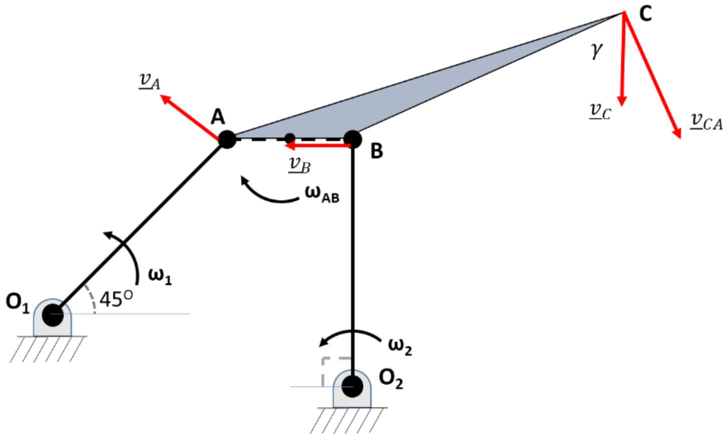
Or using Chasles' theorem

First, find angular velocity of ABC: $\underline{v}_B = \underline{v}_A + \underline{v}_{BA}$ (**+1 point** for the equation and the velocity diagram).

$v_A = \omega_1 O_1A = 30$ m/s, $v_{BA} = \omega_{AB}AB$

Perpendicular to AB (upward is positive): $v_A \sin(45°) = \omega_{AB}AB$. Hence, $\omega_{AB} = \frac{v_A \sin(45°)}{AB} = 4.24$ rad/s (**+1 point**).

Parallel to AB (left is positive): $v_A \cos(45°) = v_B = 21.21$ m/s. $\omega_2 = v_B/O_2B = 1.56$ rad/s (**+1 point**).



From ABC: BC=15.17 m, $\angle CBA = 161.6°$, AC=20.61 m, $\angle CAB = 14°$

$v_{CA} = \omega_{AB}AC = 87.39$ m/s

Finding velocity of point C: $\underline{v}_C = \underline{v}_A + \underline{v}_{CA}$

Perp. to AC (upward is +ve): $-v_C \sin(\gamma) = v_A \cos(45° - \angle CAB) - v_{CA} = -61.68$ m/s

Parallel to AC (left is +ve): $v_C \cos(\gamma) = v_A \sin(45° - \angle CAB) = 15.45$ m/s

$v_C = \sqrt{(1)^2 + (2)^2} = \sqrt{(61.68)^2 + (15.45)^2} = 63.59$ m/s (**+1 point**).

$\tan(\gamma) = 3.99, \gamma = 76°$ (this means v_C points downwards) (**+1 point**).

2) Acceleration analysis

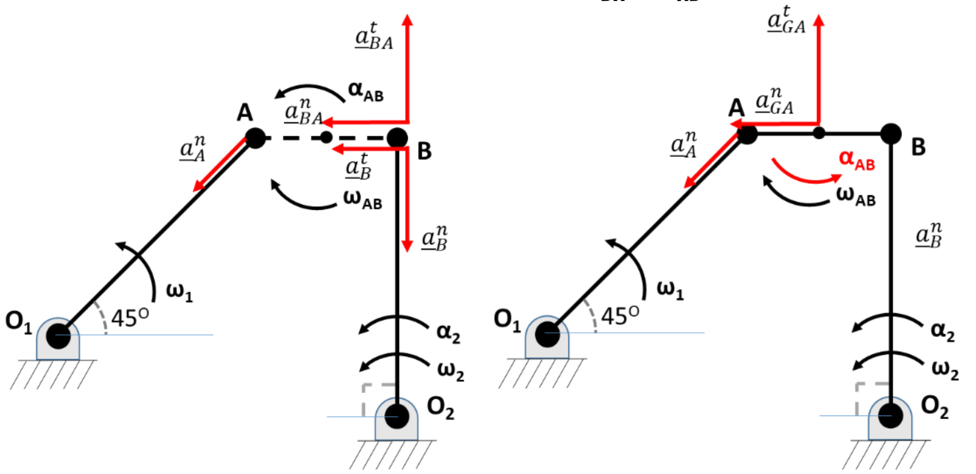
Chasles' equations for acceleration (and the sketch **+1 point**).

$$\underline{a}_B = \underline{a}_A + \underline{a}_{BA}^n + \underline{a}_{BA}^t$$
$$\underline{a}_B = \underline{a}_B^n + \underline{a}_B^t$$

Preliminary calculations:

$a_B^n = \omega_2{}^2 O_1 B = 33.12$ m/s², $a_A^n = \omega_1{}^2 O_1 A = 60$ m/s², $a_{BA}^n = \omega_{AB}{}^2 AB = 89.89$ m/s².

$$a_B^t = \alpha_2 O_2 B$$
$$a_{BA}^t = \alpha_{AB} AB$$



Perp.to AB (downwards is +ve): $a_B^n = a_A^n \sin(45°) - a_{BA}^t$.

Then finding an angular acceleration: $\alpha_{AB} = -(a_B^n - a_A^n \sin(45°))/AB = 1.86$ rad/s². (**+1 point**)

Parallel to AB (left is +ve): $a_B^t = a_A^n \cos(45°) + a_{BA}^n$

Then finding an angular acceleration: $\alpha_2 = (a_A^n \cos(45°) + a_{BA}^n)/O_2B = 9.72$ rad/s². (**+1 point**).

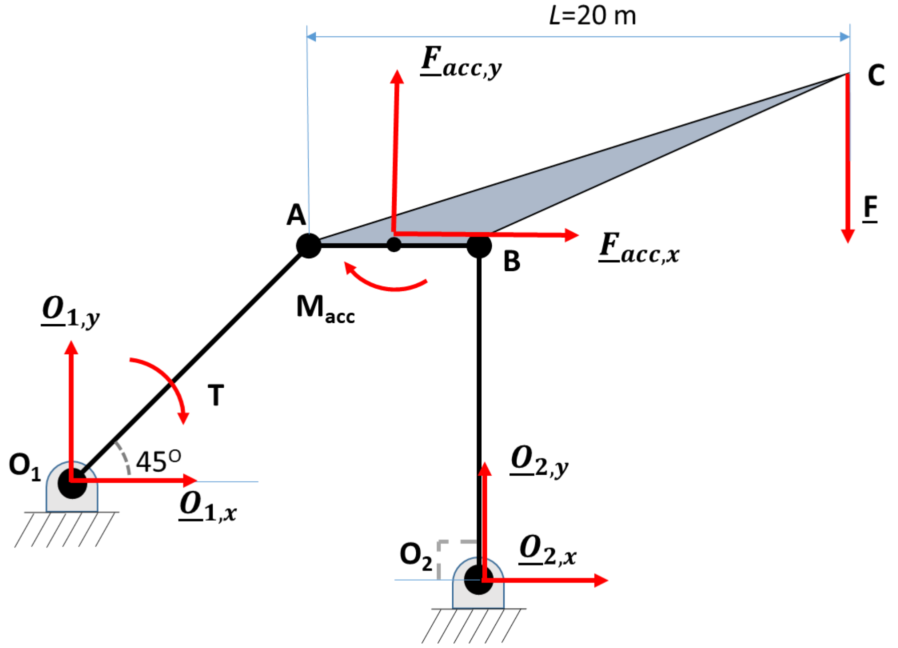
Then for point G: $\underline{a}_G = \underline{a}_A + \underline{a}_{GA}^n + \underline{a}_{GA}^t$ where $a_{GA}^n = \omega_{AB}{}^2 AG = 44.95$ m/s² and $a_{GA}^t = \alpha_{AB} AG = 4.65$ m/s². (**+1 point**)

Then, projecting on x (right is +ve) and y (upwards is +ve):

$a_{G,x} = -a_A^n \cos(45°) - a_{GA}^n = -87.37$ m/s² and $a_{G,y} = -a_A^n \sin(45°) + a_{GA}^t = -37.78$ m/s² (**+1 point with a sketch**).

3) Force analysis

FBD (+1 point per each pair of reaction forces (O2 can be simplified to vertical reaction force only), +2 point for all inertial forces/moments, +1 point for F and T) – **5 points in total for correct FBD**



From previous analysis:

$a_{G,x} = -87.37$ m/s², $a_{G,y} = -37.78$ m/s², $\alpha_{AB} = 1.86$ rad/s². M=1000 kg, J=10000 kgm2, F=1000 N.

$F_{acc,x} = 87370$ N, $F_{acc,y} = 37780$ N, $M_{acc} = 18600$ Nm.

Consider ABC + BO2:

$O_{2,x} = 0$ because O2B is massless and reaction forces should be collinear.

Take moment of ABC + O2B around point A (CCW is +ve):

$$\Sigma M_A = 0: \qquad O_{2,y} \cdot AB + F_{acc,y} \cdot \frac{AB}{2} - F \cdot L - M_{acc} = 0$$

$$O_{2,y} = \frac{F \cdot L + M_{acc} - F_{acc,y} \cdot \frac{AB}{2}}{AB} = \frac{1000 \cdot 20 + 18600 - 37380 \cdot 2.5}{5} = -11170\,N$$

(**+2 points for reactions in O2**)

Sum of all forces in Y (upwards is +ve):

$$\sum F_y = 0: \qquad O_{1,y} + F_{acc,y} + O_{2,y} - F = 0$$

$O_{1,y} = F - F_{acc,y} - O_{2,y} = 1000 - 37780 + 11170 = -25610\,N$ (**+1 point**)

Sum of all forces in X (right is +ve):

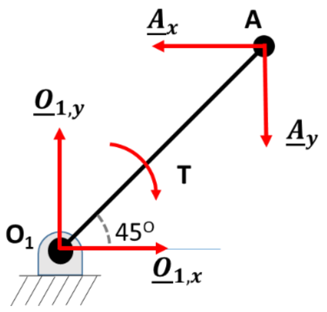
$$\sum F_x = 0: \qquad O_{1,x} + F_{acc,x} + O_{2,x} = 0$$

$O_{1,x} = -F_{acc,x} - O_{2,x} = -87370 + 0 = -87370\,N$ (**+1 point**)

Sum of moments for link O_1A around A (CCW is +ve):

$\sum M_A = 0: \qquad O_{1,x} \cdot O_1A\cos(45°) - O_{1,y} \cdot O_1A\sin(45°) - T = 0$ (**+3 points**)

$T = O_1A\frac{\sqrt{2}}{2}\left(O_{1,x} - O_{1,y}\right) = -655 \cdot 10^3 Nm.$ (**+2 point**)



Question 3:

Figure Q3 shows a system consisting of a controller with transfer function $G_c(s)$, plant with transfer function $G_p(s)$, and feedback loop with a constant proportional gain k.

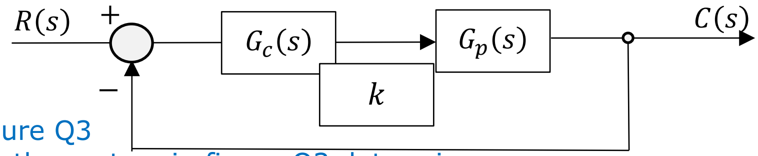


Figure Q3

For the system in figure Q3 determine:

(a) The open loop transfer function $G(s) = {C(s)}/{R(s)}$ (1 mark)

(b) The closed loop transfer function (4 marks)

Solution:

For part (a): open loop transfer function is the product of $G_c(s)$ and $G_p(s)$ so

$$G(s) = G_c(s)G_p(s)$$

1 mark for giving the correct answer

For part (b): With the feedback loop,

Step 1: $C(s) = G_c(s)G_p(s)\big(R(s) - kC(s)\big)$

Step 2: $C(s) + kG_c(s)G_p(s)C(s) = G_c(s)G_p(s)R(s)$

Step 3: $C(s)\left(1 + kG_c(s)G_p(s)\right) = G_c(s)G_p(s)R(s)$

$$G(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{\left(1 + kG_c(s)G_p(s)\right)}$$

4 marks for the correct answer, 1 marks for each step of the working as given above.

Some students may quote the result for unity feedback:

$$G(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{\left(1 + G_c(s)G_p(s)\right)}$$

This can only get a mark of 2.

The system in figure Q3 is subjected to a unit step input $R(s) = \frac{1}{s}$ at t=0. For:

$$G_p(s) = \frac{1}{(3s^2 + 10s + 8)}$$
$$G_c(s) = 3s + 4$$
$$k = 1$$

Determine:

(c) The steady state error as $t \to \infty$. (5 marks)

(d) The value of $C(t)$ at t=0.2s. (5 marks)

Solution: Part (c) For an input $R(s) = \frac{1}{s}$

$$G_c(s)G_p(s) = \frac{3s + 4}{(3s^2 + 10s + 8)} = \frac{3s + 4}{(3s + 4)(s + 2)} = \frac{1}{s + 2}$$

$$G(s) = \frac{\frac{1}{s + 2}}{\left(1 + \frac{1}{s + 2}\right)} = \frac{1}{s + 3}$$

$$C(s) = R(s)G(s) = \frac{1}{s(s + 3)}$$

From here: either

Final Value Theorem:

$$\lim_{s \to 0} s\big(R(s) - C(s)\big) = s\left(\frac{1}{s} - \frac{1}{s}\left(\frac{1}{s + 3}\right)\right) = \frac{2}{3} = 0.67\ (2sf)$$

Time domain solution:

$$C(s) = \frac{1}{s(s + 3)}$$

From table of Laplace Transforms: no. 8:

$$1 - e^{-at} \rightarrow \frac{a}{s(s+a)}$$

$$c(t) = \frac{1}{3}(1 - e^{-3t})$$

Error in the limit as $t \rightarrow \infty$ is given by

$$1 - c(\infty) = 1 - \frac{1}{3} = \frac{2}{3} = 0.67\ (2sf)$$

5 marks for the correct answer. Partial credit up to 3 marks. For the working up to $C(s) = R(s)G(s) = \frac{1}{s(s+3)}$, 2 marks, for correctly stating the final value theorem or steps to the time domain solution, 1 mark.

Solution: Part (d)

Time domain solution (if not already derived):

$$C(s) = \frac{1}{s(s+3)}$$

From table of Laplace Transforms: no. 8:

$$1 - e^{-at} \rightarrow \frac{a}{s(s+a)}$$

$$c(t) = \frac{1}{3}(1 - e^{-3t})$$

At t=0.2s,

$$c(t) = \frac{1}{3}(1 - e^{-0.6}) = 0.1504 = 0.15(2sf)$$

To eliminate the steady state error, the controller is replaced with a PID controller with the transfer function:

$$G_c(s) = \frac{s^2 + s + n}{s}$$

(e) What is the effect on the steady state error for a unit step input if $n = 1$? The plant transfer function and k remain unchanged:

$$G_p(s) = \frac{1}{(3s^2 + 10s + 8)}$$

$$k = 1$$

(5 marks)

Solution:

$$G_c(s)G_p(s) = \frac{s^2 + s + n}{s(3s^2 + 10s + 8)}$$

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} = \frac{\frac{s^2 + s + n}{s(3s^2 + 10s + 8)}}{1 + \frac{s^2 + s + n}{s(3s^2 + 10s + 8)}}$$

$$\frac{C(s)}{R(s)} = \frac{s^2 + s + n}{3s^3 + 10s^2 + 8s + s^2 + s + n} = \frac{s^2 + s + n}{3s^3 + 11s^2 + 9s + n}$$

The steady state error is therefore

$$e(t) = 1 - \lim_{s \to 0} s\big(C(s)\big) = 1 - \lim_{s \to 0} s\left(\frac{1}{s}\left(\frac{s^2 + s + n}{3s^3 + 11s^2 + 9s + n}\right)\right) = 0$$

And the PID controller will eliminate the steady state error.

(f) Derive the transfer function and draw the Routh Array for the closed loop system. At what values of k will the system be stable?

(10 marks)

Solution:

$$G_c(s)G_p(s) = \frac{s^2 + s + n}{s(3s^2 + 10s + 8)}$$

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} = \frac{\frac{s^2 + s + n}{s(3s^2 + 10s + 8)}}{1 + \frac{s^2 + s + n}{s(3s^2 + 10s + 8)}}$$

$$\frac{C(s)}{R(s)} = \frac{s^2 + s + n}{3s^3 + 10s^2 + 8s + s^2 + s + n} = \frac{s^2 + s + n}{3s^3 + 11s^2 + 9s + n}$$

$Routh\ Array$:

s^3	3	9	0
s^2	11	n	0
s	$= \frac{99 - 3n}{11}$	0	0
s^0	n	0	0

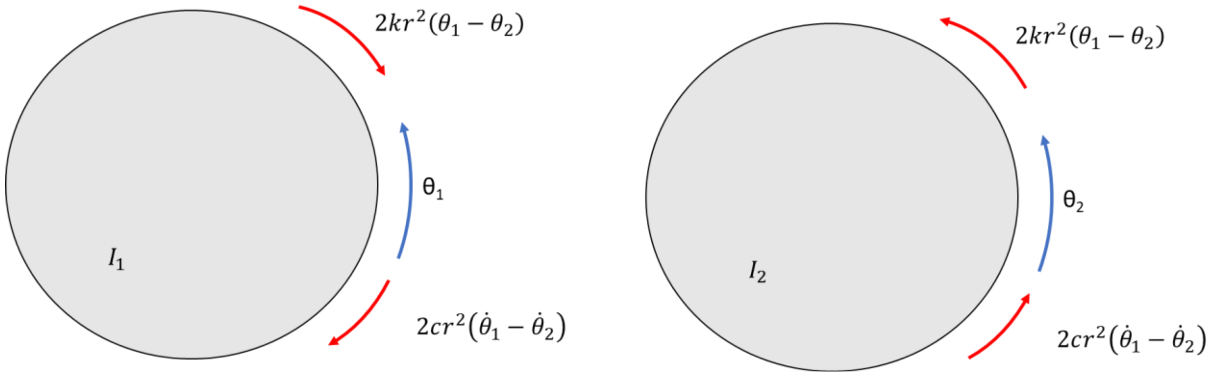
To satisfy the Routh-Hurwitz criteria, no change of sign in the denominator and all values in the first column must be positive.
Therefore the system will be stable for $0 \leq k \leq 33$ (I will also accept $0 < k < 33$ as we do not go into what to do if you have a zero in the first column)

Question 4:

A coupling is made between two rotational shafts that can be considered equivalent to the model shown in FIGURE Q4. Assume the shafts have mass moments of inertia I_1 and I_2 as shown, and have equal radii, *r*. The stiffness and damping elements in the couplings are equivalent to *k* and *c* respectively with:

r = 0.1m
k = 100N
c = 1 Ns/m
I_1 = 10 kg.m^2
I_2 = 20 kg.m^2

(a) Draw a fully annotated Free Body Diagram for each of the rotational elements in terms of their rotational displacements, θ_1 and θ_2, for the positive directions of motion shown in Figure Q3. [6 points]



Arrows and order of displacements/velocities can be reversed. Results in same EOM eventually.

3 pts each

(b) Derive the equations of motion for each of the rotational elements. [4 points]

EOM I_1

$$I_1\ddot{\theta}_1 = -2kr^2(\theta_1 - \theta_2) - 2cr^2(\dot{\theta}_1 - \dot{\theta}_2)$$

$$I_1\ddot{\theta}_1 + 2kr^2\theta_1 + 2cr^2\dot{\theta}_1 - 2kr^2\theta_2 + 2cr^2\dot{\theta}_2 = 0$$

$$10\ddot{\theta}_1 + 2\theta_1 + 0.02\dot{\theta}_1 - 2\theta_2 - 0.02\dot{\theta}_2 = 0$$

+2 points

$$I_2\ddot{\theta}_2 = 2kr^2(\theta_1 - \theta_2) + 2cr^2(\dot{\theta}_1 - \dot{\theta}_2)$$

$$I_2\ddot{\theta}_2 + 2kr^2\theta_2 + 2cr^2\dot{\theta}_2 - 2kr^2\theta_1 - 2cr^2\dot{\theta}_1 = 0$$

$$20\ddot{\theta}_2 + 2\theta_2 + 0.02\dot{\theta}_2 - 2\theta_1 - 0.02\dot{\theta}_1 = 0$$

+2 points

(c) Determine the generalized matrix, $[Z]\{\Theta\} = \{0\}$ for this system. [4 points]

$$[Z] = [K] - \omega^2[M] \text{ and } \{\Theta\} = \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}$$

From EOM we can form the matrix based EOM to help us see [K] and [M] better

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 2kr^2 & -2kr^2 \\ -2kr^2 & 2kr^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} + \begin{bmatrix} 2cr^2 & -2cr^2 \\ -2cr^2 & 2cr^2 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

2 pt

Results in

$$[Z] = \begin{bmatrix} 2kr^2 - \omega^2 I_1 & -3kr^2 \\ -2kr^2 & 2kr^2 - \omega^2 I_2 \end{bmatrix} = \begin{bmatrix} 2 - 10\omega^2 & -2 \\ -2 & 2 - 10\omega^2 \end{bmatrix}$$

2pt

(d) Solve for the undamped natural frequencies associated with this system.

[3 points]

To do this solve for the determinant

$$(2kr^2 - \omega^2 I_1)(2kr^2 - \omega^2 I_2) - (-2kr^2)(-2kr^2) = 0$$

$$200\omega^4 - 40\omega^2 = 0$$

1pt

Natural frequencies are

$$\omega_{n1} = 0$$

1pt

$$\omega_{n2} = 0.45 \frac{rad}{s} = 0.0717$$

1pt

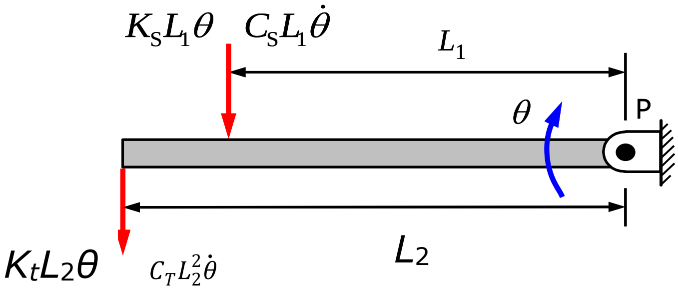
Question 5:

FIGURE Q5 shows a Single Degree of Freedom dynamic mass-spring-damper system acting through pivot, P, with damping and spring elements as shown. Data values are:

I_p = 10 kg m^2
K_t = 100000 N/m
C_t = 50 Ns/m
K_s = 20000 N/m
C_s = 1000 Ns/m
L_1 = 0.7 m
L_2 = 0.8 m

(a) Draw a fully annotated Free Body Diagram for the system. [3 points]

(a)



(b) Derive the Equation of Motion for the system in terms of rotational displacement, θ. [3 points]

Taking moments about P:

$$I_P\ddot{\theta} = -K_s L_1^2\theta - K_t L_2^2\theta - C_s L_1^2\dot{\theta} - C_T L_2^2\dot{\theta}$$

$$I_P\ddot{\theta} + K_s L_1^2\theta + K_t L_2^2\theta + C_s L_1^2\dot{\theta} + C_T L_2^2\dot{\theta} = 0$$

$$10\ddot{\theta} + 522\dot{\theta} + 73800\theta = 0$$

(c) Determine the undamped natural frequency. [2 points]

$$\omega_n = \sqrt{\frac{K_s L_1^2 + K_t L_2^2}{I_P}} = 85.9\frac{rad}{s} = 13.7Hz$$

(d) Determine the damping ratio. [2 points]

$$\zeta = \frac{C_s L_1^2 + C_T L_2^2}{2\sqrt{\left(K_s L_1^2 + K_t L_2^2\right)I_P}} = 0.3$$

(e) If the system is allowed to vibrate freely, what frequency will this occur at? [2 points]

It will occur at the damped natural frequency

$$\Omega = \omega_d = \omega_n\sqrt{1-\zeta^2} = 81.8\,\frac{rad}{s} = 13.03\ Hz$$

(f) The beam is given a push at time, t=0 s, resulting in an initial clockwise angular velocity of $\dot{\theta}_i = 6$ rad/s, from its initial rest position, $\theta_i = 0$ rad. Predict the maximum angular displacement of the axle in the subsequent free vibration. You may find the exact solution, or an approximation. If you use an approximation state any assumptions used.

[11 points]
[green points are in common]
[orange points if they use exact solution]
[purple points if they use approximate solution]

Since $\gamma = 0.304$, the system is lightly damped. The free response can be written as:

$$\theta(t) = e^{-\gamma\omega_n t}\left[B_1 \cos(\Omega_n t) + B_2 \sin(\Omega_n t)\right] \text{ where } \Omega_n = \omega_n\sqrt{1-\gamma^2}$$ [1]

From the data given, $\Omega_n = 81.85\,\text{rad/s}$

Initial conditions are: $\theta = 0$ and $\dot{\theta} = 6$ rad/s at $t = 0$

$\theta(0) = [B_1 \times 1 + B_2 \times 0] = 0$. Therefore $B_1 = 0$ [2]

Angular velocity, $\dot{\theta}(t) = B_2\left[e^{-\gamma\omega_n t}\,\Omega_n \cos(\Omega_n t) - \gamma\omega_n\, e^{-\gamma\omega_n t} \sin(\Omega_n t)\right]$

$\dot{\theta}(t) = B_2[\Omega_n \times 1 - \gamma\omega_n \times 0] = 6$ rad/s. Therefore $B_2 = \dfrac{6}{\Omega_n}$ =0.0733 [3]

Axle response is $\theta(t) = \dfrac{6}{\Omega_n} e^{-\gamma\omega_n t} \sin(\Omega_n t)$

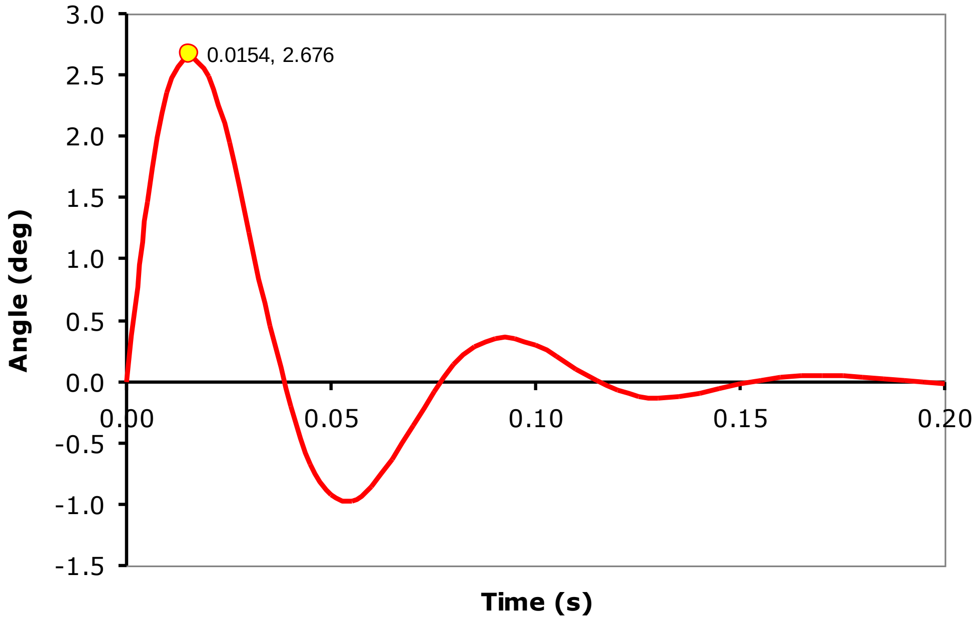
The maximum angular displacement occurs when $\dot{\theta}(t) = 0 = \Omega_n \cos(\Omega_n t) - \gamma\omega_n \sin(\Omega_n t)$

[2 if solved with exact solution]

or $\tan(\Omega_n t) = \dfrac{\Omega_n}{\gamma\omega_n}$. Hence t = 0.0154 s and θ = 0.0467 rad (2.68º).

[3 if solved with exact solution]

Not required in answer, but for marker reference only – The response waveform is:



When damping is low, the maximum response occurs approximately one quarter of a cycle after the start. Here, the period of damped vibration is 0.0768 s, implying that the maximum response will be at 0.0192 s. This gives a maximum angular displacement of 0.0444 rad (2.55º); an error of 4.9% to the actual solution.

[4 points for solution and 1 point for assumption if solved using approximate methods]

Note: The students may choose either method, but must give the assumption that maximum response occurs at approximately one quarter of a cycle after the start if using approximation methods.

[Total part e) = 11 marks]